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On the ultimate value of local time of one-dimensional super-Brownian motion

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Abstract

We study the random field of local time picked up over the entire life of a super-Brownian motion on the real line. The finite-dimensional distributions of the field are characterized via their Laplace transforms by unique solutions of certain boundary-value differential equations. In some cases the one-dimensional distributions can be found explicitly, giving some insight into how super-Brownian motion behaves before extinction or local extinction.

Keywords: Branching Brownian motion, first passage process, diffusion approximation, boundary-value differential equation, stable distribution

1. Introduction

1.1. *Purpose*

We consider in this study a class of *one-dimensional super-Brownian motions* $X = \{X_t, t \geqslant 0\}$ (SBM) on the real line $\mathbb{R}$. More specifically, using Dynkin's notation (see Dynkin, 1993), X is a $(B, K, \mathscr{A})$-superprocess on $\mathbb{R}$, where $B = \{B_t, t \geqslant 0\}$ is a standard one-dimensional Brownian motion, $K = \{K_t, t \geqslant 0\}$ is the trivial deterministic additive functional of B defined by $K_t = t$, and the function $\mathscr{A}$ is a branching exponent or parameter. It is assumed that $\mathscr{A}$ is of the form

$$\mathscr{A}(u) = \varrho u^2 + \int_0^\infty (\mathrm{e}^{-su} - 1 + su)\, \nu(ds),$$

where $\varrho \geqslant 0$ and ν is a Lèvy measure such that

$$\int_0^\infty (s \wedge s^2)\, \nu(\mathrm{d}s) < \infty.$$

These processes can be viewed as strong Markov processes with càdlàg paths and state space a set of Borel measures $\mathscr{M}$ on $\mathbb{R}$, equipped with weak or vague topologies.

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Denote integration with respect to X_t by $\langle X_t, f\rangle = \int_{\mathbb{R}} f(x)X_t(dx)$, where f is measurable, bounded and nonnegative.

A *local time process* for X at the point $a \in \mathbb{R}$ can be introduced as the limit in distribution

$$L_t^a = \lim_{\varepsilon\to 0} \int_0^t \langle X_s, f_a^\varepsilon\rangle \, ds, \qquad f_a^\varepsilon(y) = \varepsilon^{-1} f(\varepsilon^{-1}(y-a)),$$

where $f \geqslant 0$ is symmetric, smooth, has compact support and is normalized by $\int f(y)\,dy = 1$. The limit does not depend on the choice of f. The process $L^a = \{L_t^a, t \geqslant 0\}$ has been defined and studied in great detail under more general conditions than here, notably it exists also in higher dimensions. See Iscoe (1986), Fleischmann (1988), Sugitani (1989), Adler and Lewin (1992), Adler (1993) and Krone (1993). For example, it is known (with restrictions on $\mathscr{A}$) that there exists a jointly continuous version of L_t^a for $(t,a) \in [0,\infty) \times \mathbb{R}$. For the visualization of the local time functional we refer to the simulations that can be found in Adler (1994).

The aim of this study is to characterize the distribution of the *ultimate value of local time*

$$L_\infty^a := \lim_{t\to\infty} L_t^a,$$

which exists by monotone convergence. In the case X lives in the set of finite Borel measures, it is well-known that X dies out almost surely. Then the ultimate value of the local time is L_ζ^a, where ζ stands for the extinction time. However, even in the more general case of infinite initial measures it turns out that L_∞^a sometimes exists. This will shed some further light on the phenomenon of *local extinction* of superprocesses in low dimension. For example, it follows from our results in the case of continuous super-Brownian motion, i.e. $\mathscr{A}(u) = u^2$, that if $X_0 = \delta_a$, then the ultimate value of local time at a is a random variable with a one-sided stable distribution of index 2/3. Moreover, if $X_0 = dx$ (Lebesgue measure), then the ultimate value of local time at a (or any other point) is a random variable with a one-sided stable distribution of index 1/3.

Our approach to these problems is based on the superprocess *particle picture*. In fact, we construct L_∞^a, $a \in \mathbb{R}$, as a limit of the corresponding local time functionals for a sequence of *branching Brownian motions*. Such functionals have been studied in Borodin and Salminen (1993). In the course of the proof of our main result it is seen that the local time obtained in this way from the diffusion approximation coincides with the limit of L_t^a as $t \to \infty$.

1.2. Preliminaries

In order to settle notation, provide background and to prepare for the formulation of our results, we recall some aspects of branching Brownian motion and super-Brownian motion.

We start by considering a branching Brownian motion $Y = \{Y_t, t \geqslant 0\}$ (BBM) with constant creation rate κ and state independent offspring production determined by an offspring generating function $F(u) := \sum_k p_k u^k$. We assume that Y is critical, i.e. $F'(1) = 1$, and that $F(0) > 0$. Consequently, Y dies out with probability one. Suppose the BBM starts at time 0 with one particle located at x. For fixed t, Y_t is a point measure on $\mathbb{R}$,

$$Y_t = \sum_{j=1}^{N_t} \delta_{Y_t^{(j)}},$$

where N_t is the number of particles alive at t and $(Y_t^{(1)}, \ldots, Y_t^{(N_t)})$ their locations.

As a conceptual complement, it is useful to take the point of view that the sample space of Y is a collection of marked trees. We recall these notions in some detail.

A *particle* $u = (j_1, \ldots, j_n)$ is represented by a sequence of reproduction numbers $j_i \geqslant 1$ which signify that u is the j_nth child of u_{n-1}, where u_{n-1} is the j_{n-1}th child of u_{n-2}, ..., where u_2 is the j_2nd child of u_1, which is the j_1th child of the initial particle, denoted $\emptyset$. A *tree* ω is a set of particles forming a complete progeny of $\emptyset$. Hence if $u \in \omega$ all ancestors v of u ($v < u$) are in ω and also any child, grandchild, etc. of u are in ω. To each particle $u \in U$ (:= the collection of all particles) is attached a birthtime α^u, a deathtime β^u and a spatial motion $\gamma^u = \{\gamma_r^u;\ \alpha^u \leqslant r \leqslant \beta^u\}$ with $\gamma^u(\alpha^u) = 0$. Hence

$$M := \{(\alpha, \beta, \gamma);\ 0 \leqslant \alpha < \beta < \infty, \gamma : [\alpha, \beta) \mapsto \mathbb{R}, \gamma(\alpha) = 0\}$$

forms a *mark space*. The set Ω^0 of *marked trees* ω^0 is defined as a collection of elements

$$\omega^0 := \left(\omega, \{(\alpha^u, \beta^u, \gamma^u);\ u \in \omega\}\right),$$

where $(\alpha^u, \beta^u, \gamma^u) \in M$ for every $u \in \omega$, and ω runs over all trees. Let $\dagger$ denote a cemetery point for the particles. The path of a particle $u \in \omega$ whose mother is v is given by

$$\xi_t^u(\omega^0) := \begin{cases} x + \gamma_t^{\emptyset}(\omega^0), & u = \emptyset,\ 0 \leqslant t < \beta^{\emptyset}, \\ \xi_{\beta^v-}^v(\omega^0) + \gamma_t^u(\omega^0), & u \neq \emptyset,\ \alpha^u \leqslant t < \beta^u, \\ \dagger, & t \geqslant \beta^u. \end{cases}$$

The *lifelength* of u is $\beta^u - \alpha^u$. The *extinction time* of the tree is $\zeta(\omega^0) := \max_{u \in \omega} \beta^u$.

Let $\mathscr{F}^0$ denote the smallest σ-algebra on Ω^0 which makes all marked trees measurable. The set of particles born before time t in an obvious way generates a natural filtration $(\mathscr{F}_t^0)$ of $\mathscr{F}^0$, which carries the information in the marked trees up to time t. Now let P be a probability measure on the filtered space $(\Omega^0, \mathscr{F}^0, (\mathscr{F}_t^0))$ such that for each u, the lifelength $\alpha^u - \beta^u$ is exponentially distributed with parameter κ, γ^u is a standard Brownian motion starting at zero, the branching is governed by F and all particles evolve independently. The BBM is the canonical process Y on the triple $(\Omega^0, \mathscr{F}^0, \mathscr{P})$.

We can introduce the local time of Y at a point $a \in \mathbb{R}$ up to time t in two equivalent ways. First

$$\ell_t(a) := \lim_{\varepsilon \downarrow 0} \frac{1}{2\varepsilon} \int_0^t \sum_{k=1}^{N_s} 1_{(a-\varepsilon, a+\varepsilon)}(Y_s^{(k)})\, \mathrm{d}s = \lim_{\varepsilon \downarrow 0} \frac{1}{2\varepsilon} \int_0^t \langle Y_s, 1_{(a-\varepsilon, a+\varepsilon)} \rangle\, \mathrm{d}s,$$

where the limit exists a.s. Because $\zeta < \infty$ a.s., the ultimate value of the local time $\ell_\zeta(a)$ is finite at every point a.

Second, let $\ell_t^u(a)$ denote the Brownian local time at a which a particle u collects up to time t, that is

$$\ell_t^u(a) := \lim_{\varepsilon \to 0} \frac{1}{2\varepsilon} \int_{t\wedge\alpha^u}^{t\wedge\beta^u} I\{|\xi_s^u - a| < \varepsilon\}\,\mathrm{d}s.$$

Then

$$\ell_t(a) = \sum_{u\in\omega} \ell_t^u(a), \qquad \ell_\zeta(a) = \sum_{u\in\omega} \ell_{\beta^u}^u(a).$$

It is proved in Borodin and Salminen (1993, Theorem 2.2), that the Laplace function

$$x \mapsto E_x \exp\left\{ -\sum_{i=1}^k \lambda_i \ell_\infty(a_i) \right\},$$

is the unique solution of the elastic boundary-value problem

$$\begin{aligned} &\frac{1}{2} g'' = -\kappa\big(F(g) - g\big) && \text{on } \mathbb{R} \setminus \{a_1, \ldots, a_k\} \\ &g'(a_i+) - g'(a_i-) = 2\lambda_i g(a_i), && i = 1, \ldots, k, \\ &\lim_{x\to\pm\infty} g(x) = 1, \\ &0 < g \leqslant 1. \end{aligned} \tag{1.1}$$

Returning to the SBM we recall the following facts; consult e.g. Dawson (1993) for a full account. The state space of X is either the set $\mathscr{M}^1$ of finite measures equipped with the weak topology or a set $\mathscr{M}^p$, $p > 1$, of tempered measures furnished with the p-vague topology. The latter are such that $\langle \mu, \psi_p \rangle < \infty$, where $\psi_p(x) = 1/(1+|x|^2)^{p/2}$. We write $\mathscr{C}$ for the corresponding "dual" space of nonnegative continuous functions. In the finite-measure case, $\mathscr{C}$ is simply the set of bounded, continuous, nonnegative functions on $\mathbb{R}$. In the p-vague case $\mathscr{C}$ is the set of continuous, nonnegative functions ψ on $\mathbb{R}$ such that $\psi(x)/\psi_p(x)$ has a limit as $|x| \to \infty$ and $\|\psi/\psi_p\| < \infty$, where $\|\cdot\|$ is the supremum norm. Fix $t > 0$ and put

$$V_t(x) := -\log \boldsymbol{E}_{\delta_x} \exp\langle X_t, -\varphi \rangle, \quad t \geqslant 0, \quad \varphi \in \mathscr{C},$$

where $\boldsymbol{E}_{\delta_x}$ denotes the expectation operator for the law $\boldsymbol{P}_{\delta_x}$ of X given that $X_0 = \delta_x$. Let $p(t; x, y)$ denote the transition probability density for B and put

$$S_t\varphi(x) := E_x^B \varphi(B_t) = \int p(t; x, y)\varphi(y)\,\mathrm{d}y.$$

The SBM can be defined by the fact that V is the unique positive solution of the integral equation

$$V_t(x) = S_t\varphi(x) - \int_0^t S_{t-r}\mathscr{A}\big(V_r\big)(x)\,\mathrm{d}r, \quad t \geqslant 0.$$

For an arbitrary initial measure $\mu \in \mathscr{M}$ at time $t = 0$, the more general relation

$$-\log \boldsymbol{E}_\mu \exp\langle X_t, -\varphi \rangle = \langle \mu, V_t \rangle$$

is called the *branching property* of X. The finite-dimensional distributions of X are obtained by an iterative application of the branching property, see e.g. Dawson (1993, Corollary 4.4.6).

Turning to the local time of X, the log-Laplace functions

$$v_t(x) := -\log \boldsymbol{E}_{\delta_x} \exp\left\{ -\sum_{i=1}^{k} \lambda_i L_t^{a_i} \right\}$$

are the unique positive solutions to

$$v_t(x) = \sum_{i=1}^{k} \lambda_i \int_0^t p(r; x, a_i)\,\mathrm{d}r - \int_0^t \int_{-\infty}^{\infty} p(t-r; x, y) \mathscr{A}\big(v_r(y)\big)\,\mathrm{d}y\,\mathrm{d}r\,. \tag{1.2}$$

One may also consider (1.2) as the defining relation for v_t to be the solution of the parabolic initial-value problem of the generalized form

$$\begin{aligned} \frac{\partial v_t}{\partial t} &= \frac{1}{2}\frac{\partial^2 v_t}{\partial x^2} - \mathscr{A}(v_t) + \sum_{i=1}^{k} \lambda_i\, \delta_{a_i}\,, \\ v_0(x) &= 0\,, \end{aligned}$$

compare Fleischmann (1988). Our approach on the contrary is based on true differential equations where the parameters $\lambda_1, \ldots, \lambda_k$ enter into the boundary conditions.

Note that for the study of L_∞^a, we cannot proceed in the straightforward manner letting $t \to \infty$ in (1.2) since both integrals on the right side diverge.

It remains to describe the basic connection between X and Y. For a given branching exponent $\mathscr{A}$ consider a sequence $Y^{(n)}$, $n \geqslant 1$, where each $Y^{(n)}$ is a BBM with creation rate $\kappa_n = \mathscr{A}'(n)$ and (critical) offspring generating function $F_n(u) = u + \mathscr{A}(n(1-u))/n\kappa_n$. Then

$$\mathscr{A}(u) = n\kappa_n\big(F_n(1-u/n) - (1-u/n)\big)\,. \tag{1.3}$$

Assume the initial condition $Y_0^{(n)} = n\delta_x$, i.e. we have n independent particles at x. This sequence yields the SBM with branching exponent $\mathscr{A}$ as the weak limit,

$$\frac{1}{n} Y^{(n)} \Longrightarrow X\big(\text{weak convergence in Skorokhod topology on } \mathscr{D}([0,\infty), \mathscr{M}^1)\big)\,. \tag{1.4}$$

This choice of Y^n is, perhaps, the simplest one in order to construct a specific SBM. Under appropriate assumptions more general sequences κ_n and F_n also give rise to the same SBMs. A typical condition is that for all $m > 0$ and $n \geqslant m$

$$\lim_{n\to\infty} \sup_{u \leqslant m} \left| n\mathscr{A}_n\left(1 - \frac{u}{n}\right) - \mathscr{A}(u)\right| = 0\,. \tag{1.5}$$

where $\mathscr{A}_n(u) := \kappa_n(F_n(u) - u)$.

It is known in the case $\mathscr{A}(u) = u^2$ and for fixed a and fixed $t > 0$ (see Adler, 1993, Theorem 2.3) that the BBM local time $\ell_t^{(n)}(a)$ of $Y^{(n)}$ converges weakly to the SBM local time L_t^a of X as $n \to \infty$. In Section 4.2 we need and prove an extension of this result to the case with an arbitrary number of points and general branching exponent $\mathscr{A}(u)$.

1.3. Plan of the paper

The main result is formulated in Section 2. We give therein also a corollary containing some explicit formulae for the distribution of L^a_∞. Sections 3 and 4 are devoted to the proof of the main result, considering the case $k = 1$ in Section 3 and the general case in Section 4. More specifically, in Section 3 we prove the existence and characterization of the limit distribution of the particle system local time at one point. However, we stress that the results on the case $k = 1$ are *not* needed in the general proof. Section 3 is included because to us this proof seems interesting and rather simple in comparison with the more laborious approximation method applied in the general case. When $k = 1$ the idea is to use a link between local time and the number of first passages of particles in the branching Brownian motion. Indeed, the local time at one point can be decomposed into independent components corresponding to subtrees starting from the first passage particles. Some properties of the first passage process, studied in Kaj and Salminen (1993a, b), then lead to the desired result. For $k > 1$ the distributions exhibit rather complicated dependencies which, perhaps, make such an approach impossible in general.

Therefore, in Section 4 we adopt another strategy. Note that one difficulty here is that the quantities L^a_∞ do not possess finite mean values. Because of this there does not seem to exist an integral equation from which the law of L^a_∞ can be found. Nevertheless we can begin with a differential problem related to the particle systems. From there we derive a uniform bound on the joint log-Laplace function of the approximating vector in (2.1) below, which we denote by $u^{(n)}$. This immediately gives a uniform bound on the corresponding finite time quantity $u^{(n)}_t$. By the usual Gronwall-type argument this carries over to the solutions v_t of (1.2). Again turning to differential equations we then show that $v = \lim_{t\to\infty} v_t$ is the solution of the differential problem (2.3) stated in the theorem below.

However, in order to prove the theorem we must apply both the finite time approximation and the scheme of particle systems approximation in a simultaneous manner. To describe the method recall that $u^{(n)}$ and v are now the log-Laplace functions of the left and the right side of (2.1), respectively. The proof that v is the limit of $u^{(n)}$ will be based on the inequality

$$\|u^{(n)} - v\| \leqslant \|u^{(n)} - u^{(n)}_{t_n}\| + \|u^{(n)}_{t_n} - v_{t_n}\| + \|v_{t_n} - v\|, \tag{1.6}$$

finding a scheme (n, t_n) such that the right side converges to zero. The three terms on the right side are successively the topics of Sections 4.1, 4.2 and 4.3.

2. Limit theorem

Our main result is the following

Theorem. *Suppose $X_0 = \delta_x$ and consider the local time random field defined by $L^a_\infty = \lim_{t\to\infty} L^a_t$, $a \in \mathbb{R}$. The finite-dimensional distributions of the corresponding* *scaled local times of $Y^{(n)}$ converge to those of L^a_∞:*

$$\frac{1}{n}\left(\ell^{(n)}_\infty(a_1),\dots,\ell^{(n)}_\infty(a_k)\right) \xrightarrow{\mathrm{d}} \left(L^{a_1}_\infty,\dots,L^{a_k}_\infty\right), \tag{2.1}$$

for any k and $a_1,\dots,a_k$. The joint log-Laplace function

$$v(x) = -\log \boldsymbol{E}\Big[\exp - \sum_{i=1}^{k} \lambda_i L^{a_i}_\infty \,|\, X_0 = \delta_x\Big], \quad \lambda_1,\dots,\lambda_k > 0, \tag{2.2}$$

of the limit distribution is the unique solution of the differential boundary-value problem

$$\begin{aligned} &\frac{1}{2}v'' = \mathscr{A}(v) && \text{on } \mathbb{R}\setminus\{a_1,\dots,a_k\},\\ &v'(a_i-) - v'(a_i+) = 2\lambda_i, && i = 1,\dots,k,\\ &\lim_{x\to\pm\infty} v(x) = 0,\\ &v \geqslant 0. \end{aligned} \tag{2.3}$$

The solution v and its first and second derivatives are uniformly bounded on $\mathbb{R}$.

The claim of the example in Section 1.1 for continuous SBM is immediate by taking $\beta = 1$ and $\mu = \delta_a$ or $\mu = \mathrm{d}x$ in the following

Corollary. *Restrict to the case $\mathscr{A}(u) = u^{1+\beta}$, $0 < \beta \leqslant 1$. For a fixed such β, if $X_0 = \mu$ belongs to $\mathscr{M}^p$ for some $1 \leqslant p \leqslant 2/\beta$ then the ultimate value L_∞ of the local time of X exists. The one-dimensional distributions of L_∞ are then given by*

$$-\log \boldsymbol{E}_\mu \exp -\lambda L^a_\infty = \langle \mu, v(a-\cdot)\rangle, \quad a \in \mathbb{R},$$

where

$$v(x) = \frac{\lambda^{2/(2+\beta)}}{\left(\left(2/\sqrt{2+\beta}\right)^{\beta/(2+\beta)} + \beta\lambda^{\beta/(2+\beta)}\,|x|/\sqrt{2+\beta}\right)^{2/\beta}}, \quad \lambda > 0.$$

In particular, if $\mu = dx$ then for each $1 < \beta \leqslant 1$

$$\langle \mu, v\rangle = \int v(x)\,\mathrm{d}x = \frac{(2+\beta)^{1/\beta}}{4^{(1-\beta)/\beta}(2-\beta)}\,\lambda^{(2-\beta)/(2+\beta)}.$$

Proof. In this case, for $x < a_1$,

$$\begin{aligned} \tfrac{1}{2}\left(v'(x)\right)^2 &= \textstyle\int_{-\infty}^{x} v'(y)v''(y)\,\mathrm{d}y = 2\int_{-\infty}^{x} v'(y)v(y)^{1+\beta}\,\mathrm{d}y\\ &= 2\textstyle\int_0^{v(x)} s^{1+\beta}\,\mathrm{d}s = 2v(x)^{2+\beta}/(2+\beta). \end{aligned}$$

Hence $v'(x) = c\,v(x)^{1+\beta/2}$ for some constant c. Use the boundary condition at a_1 together with the bound on $|v'(a_1+)|$ to see that $v(x)$ is asymptotically $\sim |x|^{-2/\beta}$ as $x \to -\infty$. By symmetry, this also holds for $|x| \to \infty$. Hence $v \in \mathscr{C}$ for all $p \leqslant 2/\beta$ and so $\langle \mu, v\rangle < \infty$ if $\mu \in \mathscr{M}^p$ for some $1 \leqslant p \leqslant 2/\beta$.

Now take $k = 1$ and $a = 0$, say. By symmetry the boundary condition at 0 simplifies and we only have to solve the equation

$$v'(x) = c_\beta\, v(x)^{1+\beta/2}, \quad v'(0) = \lambda.$$

Taking the form of the constants c_β into consideration, straightforward calculations result in the formulas stated in the corollary. □

3. One-dimensional distributions

In this section we prove when $k = 1$ that the limit in (2.1) exists and is characterized by (2.3). It was already mentioned that the special approach we use here is based on the diffusion approximation result for the first passage process presented in Kaj and Salminen (1993a). We begin with some introductory facts needed in the proof.

3.1. Background

Consider the BBM process Y and its local time $\ell_\infty(a)$. Without loss of generality we take $a = 0$ and set $\ell_\infty := \ell_\infty(0)$. For $\lambda > 0$ introduce the function

$$\psi(x, \lambda) := E_x\left[\exp -\lambda\, \ell_\infty\right].$$

It follows from (1.1) that for $x \neq 0$ the function $\psi := \psi(\cdot, \lambda)$ is a solution of the differential equation

$$\tfrac{1}{2} g'' + \kappa(F(g) - g) = 0.$$

Let $x < 0$, multiply with ψ', integrate from $-\infty$ to x and change variables to see that ψ satisfies

$$\left(\psi'(x)\right)^2 = 4\kappa \int_{\psi(x)}^{1} (F(u) - u)\, du. \tag{3.1}$$

By symmetry, (3.1) holds also for $x > 0$. Use the elastic boundary condition in (1.1) at $a = 0$, relation (3.1) and the continuity of ψ to get

$$2\lambda\psi(0) = \psi'(0+) - \psi'(0-) = 2\sqrt{4\kappa \int_{\psi(0)}^{1} (F(u) - u)\, \mathrm{d}u},$$

that is,

$$\psi^2(0) = \frac{4\kappa}{\lambda^2} \int_{\psi(0)}^{1} (F(u) - u)\, \mathrm{d}u, \tag{3.2}$$

which determines $\psi(0)$ uniquely. The function ψ can then be obtained by solving (3.1) with $\psi(0)$ as the initial value.

We take another look at the function ψ via the first passage process $Z = \{Z_x;\, x \in \mathbb{R}\}$. Assuming that Y starts with one particle located at 0 at time 0, Z_x is the number of particles in the whole realization of Y which visit x but do not have an ancestor who has visited x. We set $Z_0 = 1$. The process $Z^+ := \{Z_x, x \geqslant 0\}$ is a continuous-time Galton–Watson process such that the function

$$\phi(x,s) := E_0\left[s^{Z_x}\right], \quad 0 \leqslant s \leqslant 1,$$

is the unique continuous solution of the initial value problem

$$\begin{aligned} g'(x) &= \left(4\kappa \int_{g(x)}^{1} (F(u) - u)\,\mathrm{d}u\right)^{1/2}, \\ g(0) &= s, \end{aligned} \tag{3.3}$$

see Kaj and Salminen (1993a, Theorem 1). The connection between the local time and the first passage process is as follows. Given $Y_0 = \delta_x$, the local time at zero consists of a sum of Z_x independent contributions. Each of them is the local time picked up at zero in the subtree starting at zero at the first passage time of one of the first passage particles. Now the fact that ϕ satisfies (3.3) is also obvious from the corresponding property of ψ because, by the spatial homogeneity,

$$\begin{aligned} \psi(x) &= E_x[\exp -\lambda \ell_\infty] = E_0[\exp -\lambda \ell_\infty(x)] \\ &= E_0\left[\left(E_x[\exp -\lambda \ell_\infty(x)]\right)^{Z_x}\right] \\ &= E_0\left[\psi(0)^{Z_x}\right] = \phi(x, \psi(0)). \end{aligned} \tag{3.4}$$

Since the limiting behaviour of Z^+ is known under the diffusion approximation, see below, the simple relationship (3.4) is the key in finding the limiting distribution of ℓ_∞.

3.2. Existence and characterization in the case $k = 1$

Let $E^{(n)}$ refer to expectation for the BBM $Y^{(n)}$. For simplicity take κ_n and F_n such that (1.3) holds. Since the results for Z we are going to quote hold in the more general situation we could also work under assumption (1.5). Put

$$\psi^{(n)}(x,\lambda) := E_x^{(n)}\left[\exp -\lambda \ell_\infty\right], \qquad u^{(n)}(x) := n\left(1 - \psi^{(n)}(x, \lambda/n)\right).$$

We obtain from (3.2) substituting $v = n(1-u)$

$$\begin{aligned} \left(\psi^{(n)}(0,\lambda/n)\right)^2 &= \frac{4\kappa_n}{(\lambda/n)^2} \int_{\psi^{(n)}(0,\lambda/n)}^{1} (F_n(u) - u)\,\mathrm{d}u \\ &= \frac{4}{\lambda^2} \int_0^{u^{(n)}(0)} \mathscr{A}(v)\,\mathrm{d}v. \end{aligned}$$

(It is clear from the last equality that under assumption (1.5) a remainder term appears at this point. It is simple to see that it will vanish and not affect the idea of the proof.) Assume now that there exists a sequence $\{n_k\}$ such that

$$\lim_{k\to\infty} u^{(n_k)}(0) = +\infty.$$

Let $N > 0$ be given and choose k_N so large that for all $k > k_N$ it holds $u^{(n_k)}(0) > N$. Then for $k > k_N$

$$1 \geqslant \left(\psi^{(n_k)}(x, \lambda/n_k)\right)^2 \geqslant \frac{4}{\lambda^2} \int_0^N \mathscr{A}(v)\,\mathrm{d}v.$$

Because N is arbitrary and $v \to \mathscr{A}(v)$ is increasing we have reached a contradiction, that is, $\limsup_{n\to\infty} u^{(n)}(0) < \infty$. Assume next that

$$v_* := \liminf_{n\to\infty} u^{(n)}(0) < \limsup_{n\to\infty} u^{(n)}(0) =: v^* ,$$

and let $\{m_k\}$ and $\{n_k\}$ be two sequences such that, respectively,

$$\lim_{k\to\infty} u^{(m_k)}(0) = v_* \geqslant 0, \qquad \lim_{k\to\infty} u^{(n_k)}(0) = v^* < \infty .$$

Clearly,

$$\lim_{k\to\infty} \psi^{(m_k)}(0, \lambda/m_k) = \lim_{k\to\infty} \psi^{(n_k)}(0, \lambda/n_k) = 1 ,$$

and it follows

$$1 = \frac{4}{\lambda^2} \int_0^{v^*} \mathscr{A}(v)\mathrm{d}v > \frac{4}{\lambda^2} \int_0^{v_*} \mathscr{A}(v)\mathrm{d}v = 1 .$$

This contradiction shows that $v_* = v^*$. Moreover, $v(0) := \lim u^{(n)}(0)$ is the unique solution of the equation

$$1 = \frac{4}{\lambda^2} \int_0^{v(0)} \mathscr{A}(v)\,\mathrm{d}v. \tag{3.5}$$

Next let $x > 0$, and argue as in (3.4) to get

$$\psi^{(n)}(x, \lambda/n) = E^{(n)}\left[\left(\psi^{(n)}(0, \gamma/n)\right)^{Z_x}\right] = E^{(n)}\left[\exp\left\{ - \frac{\widehat{u}^{(n)}(0)}{n} Z_x\right\}\right],$$

where

$$\widehat{u}^{(n)}(0) := -n \log \psi^{(n)}(0, \lambda/n).$$

Clearly, $\lim \widehat{u}^{(n)}(0) = \lim u^{(n)}(0) = v(0)$. Let $\varepsilon > 0$ be given and choose n_ε so that for all $n > n_\varepsilon$

$$v(0) - \varepsilon \leqslant \widehat{u}^{(n)}(0) \leqslant v(0) + \varepsilon .$$

Then

$$E^{(n)} \exp\left\{ - \frac{v(0)+\varepsilon}{n} Z_x\right\} \leqslant \psi^{(n)}(x, \frac{\lambda}{n}) \leqslant E^{(n)} \exp\left\{ - \frac{v(0)-\varepsilon}{n} Z_x\right\},$$

and, consequently,

$$\begin{aligned} -n \log E^{(n)} \exp\left\{ - \frac{v(0)+\varepsilon}{n} Z_x\right\} &\geqslant - n \log \psi^{(n)}(x, \frac{\lambda}{n}) \\ &\geqslant - n \log E^{(n)} \exp\left\{ - \frac{v(0)-\varepsilon}{n} Z_x\right\}. \end{aligned}$$

In Kaj and Salminen (1993a, Theorem 2), it is proved that

$$h(x) := \lim_{n\to\infty} -n \log E^{(n)} \exp\left\{ - \frac{\beta}{n} Z_x\right\}$$

exists and is the unique continuous positive solution of the initial-value problem

$$\begin{aligned} h'(x) &= -2\Big(\int_0^{h(x)} \mathscr{A}(u)\,du\Big)^{1/2}, \\ h(0) &= \beta > 0. \end{aligned} \tag{3.6}$$

Therefore, we have

$$h_+(x) \geqslant \limsup_{n\to\infty} -n\log\psi^{(n)}\left(x,\frac{\lambda}{n}\right) \geqslant \liminf_{n\to\infty} -n\log\psi^{(n)}\left(x,\frac{\lambda}{n}\right) \geqslant h_-(x),$$

where $h_\pm$ are the solutions of (3.6) with the initial values $v(0)\pm\varepsilon$, respectively. Since the solutions are continuous and exist globally it is obvious that for every fixed $x > 0$,

$$\lim_{\varepsilon\to 0} h_+(x) = \lim_{\varepsilon\to 0} h_-(x).$$

Hence for $x > 0$,

$$v(x) := \lim_{n\to\infty} -n\log\psi^{(n)}\left(x,\frac{\lambda}{n}\right)$$

exists and satisfies (3.6), therefore also $v''(x) = 2\mathscr{A}(v(x))$, with the initial value $v(0)$ determined from (3.5). Simple arguments show further that

$$\lim_{x\to\pm\infty} v(x) = 0.$$

To check the condition for the first derivatives at 0, note that using (3.6) and (3.5) we get

$$v'(0+) = -2\Big(\int_0^{v(0)} \mathscr{A}(u)\,\mathrm{d}u\Big)^{1/2} = -\lambda.$$

The case $x < 0$ can be treated analogously, and we have

$$v'(0-) = 2\Big(\int_0^{v(0)} \mathscr{A}(u)\,\mathrm{d}u\Big)^{1/2} = \lambda.$$

These two equalities give the desired condition, and the proof is complete. □

4. Finite-dimensional distributions of L_∞

Recall (1.6), where v_t and v are the log-Laplace functions in (1.2) and (2.2), and $u^{(n)}$ and $u_t^{(n)}$ are defined formally in (4.1) and (4.5) below. The first term on the right side of (1.6) concerns the amount of mass left in the branches of the tree which have survived up to time t. The second term can be analysed by means of a Gronwall type inequality and the third term we study by taking the limit in a differential system.

4.1. Estimates based on the tree structure

Consider the basic sequence $Y^{(n)}$ of BBMs with $\kappa_n = \mathscr{A}'(n)$ and $F_n(u) = u + \mathscr{A}(n(1-u))/n\kappa_n$ as in (1.3). Recall that $E^{(n)}$ denotes expectation for $Y^{(n)}$ and define

$$u^{(n)}(x) := n\Big(1 - E_x^{(n)} \exp\Big\{-\frac{1}{n}\sum_{i=1}^{k} \lambda_i \ell_\infty(a_i)\Big\}\Big). \tag{4.1}$$

Lemma 4.1. *Fix an integer k and a partition of the (extended) real line $-\infty = a_0 < a_1 < \ldots < a_k < a_{k+1} = \infty$. For each fixed n, the function $u^{(n)}$ is the unique solution of the system*

$$\begin{aligned}
&\tfrac{1}{2}g'' = \mathscr{A}(g) && \text{on } \mathbb{R}\setminus\{a_1,\ldots,a_k\},\\
&g'(a_i-) - g'(a_i+) = 2\lambda_i\big(1 - g(a_i)/n\big), && i = 1,\ldots,k,\\
&\lim_{x\to\pm\infty} g(x) = 0,\\
&g \geqslant 0.
\end{aligned}$$

Moreover, there is a constant C such that for any n,

$$\|u^{(n)}\| \leqslant C. \tag{4.2}$$

Proof. By (1.1),

$$x \mapsto E_x^{(n)} \exp\Big\{-\frac{1}{n}\sum_{i=1}^{k} \lambda_i \ell_\infty(a_i)\Big\}$$

is the unique solution of the boundary-value problem

$$\begin{aligned}
&\tfrac{1}{2}g'' = \kappa_n\big(F_n(g) - g\big) && \text{on } \mathbb{R}\setminus\{a_1,\ldots,a_k\}\\
&g'(a_i+) - g'(a_i-) = 2\lambda_i\, g(a_i)/n, && i = 1,\ldots,k,\\
&\lim_{x\to\pm\infty} g(x) = 1,\\
&0 < g \leqslant 1.
\end{aligned}$$

Rescale according to (4.1) and use (1.3) to obtain the system in the lemma.

We turn to the uniform bound of the solutions $u^{(n)}$. Suppose g is a solution of the differential problem in the lemma. We show first

$$|g'(a_i\pm)| \leqslant \sum_{j=1}^{k} \lambda_j, \quad i = 1,\ldots,k. \tag{4.3}$$

Since $g(-\infty) = 0$ and $g'' \geqslant 0$, the first derivative g' is nonnegative on $(-\infty, a_1]$. Also, by the convexity of g on (a_i, a_{i+1})

$$g'(a_{i+1}-) - g'(a_i+) = \textstyle\int_{a_i}^{a_{i+1}} g''(y)\,\mathrm{d}y \geqslant 0, \quad i = 0,\ldots,k,$$

where $g'(-\infty) = g'(\infty) = 0$ is used. This and the relations $g'(a_i-) - g'(a_i+) = \lambda_i(1 - g(a_i)/n)$ yield

$$\begin{aligned} 0 \leqslant g'(a_1-) &= \lambda_1(1 - g(a_1)/n) + g'(a_1+) \leqslant \lambda_1(1 - g(a_1)/n) + g'(a_2-) = \cdots \\ &\leqslant \lambda_1(1 - g(a_1)/n) + \ldots + \lambda_{i-1}(1 - g(a_{i-1})/n) + g'(a_i-) = \cdots \\ &\leqslant \sum_{j=1}^{k} \lambda_j(1 - g(a_j)/n) \leqslant \sum_{j=1}^{k} \lambda_j . \end{aligned} \tag{4.4}$$

By a symmetric argument going through the points a_i in reverse order we have also

$$\begin{aligned} 0 \leqslant -g'(a_k+) \leqslant \lambda_k - g'(a_k-) \leqslant \lambda_k - g'(a_{k-1}+) \leqslant \ldots \\ \leqslant \lambda_k + \ldots + \lambda_{i+1} - g'(a_i+) \leqslant \ldots \leqslant \sum_{j=1}^{k} \lambda_j . \end{aligned}$$

It is easy to see that these inequalities and (4.4) imply (4.3). Since $|g'|$ takes its maximum at a point a_i, we even have $\sup_x |g'(x)| \leqslant C$. But a continuous nonnegative function g with $g(\pm\infty) = 0$ which has piecewise uniformly bounded derivatives must itself be uniformly bounded on $\mathbb{R}$. $\square$

Now define

$$u_t^{(n)}(x) := n\Big(1 - E_x^{(n)} \exp\Big\{ -\frac{1}{n} \sum_{i=1}^{k} \lambda_i \ell_t(a_i)\Big\}\Big). \tag{4.5}$$

By construction, $u_t^{(n)} \leqslant u^{(n)}$ for all $t > 0$, hence by (4.2)

$$\|u_t^{(n)}\| \leqslant C. \tag{4.6}$$

Lemma 4.2. *For any sequence $t_n > 0$ such that $t_n \to \infty$ as $n \to \infty$,*

$$\lim_{n\to\infty} \|u^{(n)} - u_{t_n}^{(n)}\| = 0.$$

Proof. We consider $Y^{(n)}$ and $Y_t^{(n)}$ under measure $P_x^{(n)}$, the law of the BBM starting with a single particle at x, in the same canonical sample space $(\Omega^0, \mathscr{F}^0)$. Denote by

$$K_t(\omega^0) := \{u \in \omega\,;\ \alpha^u(\omega^0) < t \leqslant \beta^u(\omega^0)\},$$

the set of particles alive at time t. Clearly $Y_t^{(n)}$ is obtained trajectorywise from $Y^{(n)}$ by cutting the tree at the space–time points (ξ_t^u, t), for all $u \in K_t$. Also, $N_t = |K_t|$ is the number of branches cut off. Now

$$\begin{aligned} &|u^{(n)}(x) - u_t^{(n)}(x)| \\ &= n\,E_x^{(n)}\Big[\exp\Big\{-\frac{1}{n}\sum_{i=1}^{k}\lambda_i\ell_t(a_i,\omega^0)\Big\} - \exp\Big\{-\frac{1}{n}\sum_{i=1}^{k}\lambda_i\ell_\infty(a_i,\omega^0)\Big\}\Big] \\ &\leqslant n\,E_x^{(n)}\Big[1 - \exp\Big\{-\frac{1}{n}\sum_{u\in K_t(\omega^0)}\sum_{i=1}^{k}\lambda_i\ell_\infty(a_i,\omega^0\circ\theta_t^u)\Big\}\Big]. \end{aligned}$$

Here the shift operator θ_t^u maps ω^0 to the marked subtree of ω^0 which starts from the particle u at time t, see Chauvin (1991) for a rigorous definition. To proceed we use the fact that given the starting points ξ_t^u, $u \in K_t$, the corresponding subtrees are independent. Therefore,

$$|u^{(n)}(x) - u_t^{(n)}(x)| \leqslant n E_x^{(n)}\Big[1 - \prod_{u\in K_t(\omega^0)} \exp\Big\{ -\frac{1}{n}\sum_{i=1}^k \lambda_i \ell_\infty(a_i, \omega^0 \circ \theta_t^u)\Big\}\Big]$$
$$= n E_x^{(n)}\Big[1 - \prod_{u\in K_t} E_{\xi_t^u}^{(n)} \exp\Big\{ -\frac{1}{n}\sum_{i=1}^k \lambda_i \ell_\infty(a_i)\Big\}\Big]$$
$$= n E_x^{(n)}\Big[1 - \prod_{u\in K_t} \big(1 - u^{(n)}(\xi_t^u)/n\big)\Big].$$

By (4.2), $u^{(n)}$ is uniformly bounded by a constant C. Hence

$$|u^{(n)}(x) - u_t^{(n)}(x)| \leqslant n E_x^{(n)}\Big[1 - \big(1 - C/n\big)^{N_t}\Big] =: h_n(t),$$

where the right hand side is in fact independent of x. The generating function $h(t) = Es^{N_t}$ for the unscaled Galton–Watson process N_t satisfies the equation

$$h'(t) = -\kappa\big(F(h(t)) - h(t)\big), \quad h(0) = s.$$

Hence, according to (1.3)

$$h_n'(t) = n\kappa_n\big(F_n(1 - h_n(t)/n) - (1 - h_n(t)/n)\big) = \mathscr{A}(h_n(t)), \quad h_n(0) = C.$$

But this implies that $h_n(t)$ is in fact also independent of n and equals the log-Laplace function of the continuous state branching process $|X_t|$ given by the total mass of the corresponding SBM X. Hence for any sequence $\{t_n\}$,

$$|u^{(n)}(x) - u_{t_n}^{(n)}(x)| \leqslant -\log \boldsymbol{E} \exp -C\,|X_{t_n}|.$$

Since X dies out almost surely the right side goes to zero when $t_n \to \infty$ as $n \to \infty$. □

4.2. *Diffusion approximation of BBM*

We continue the study of the function $u_t^{(n)}$. In this section we estimate the rate of convergence to zero for the difference $\|u_{t_n}^{(n)} - v_{t_n}\|$.

We start by deriving an integral equation for the Laplace functions of ℓ_t under the measure P_x. Such equations are basic tools for the study of superprocesses, see e.g. Dawson (1993, Lemma 4.3.4), or Dynkin (1993, Lemma 1.5). However, we feel it is worthwhile to give a direct proof of the version we will use here even if very similar equations are covered in the literature.

Lemma 4.3. *The function*

$$w_t(x) := E_x \exp\Big\{ -\sum_{i=1}^k \lambda_i \ell_t(a_i)\Big\}$$

satisfies the integral equation

$$\begin{aligned} w_t(x) &= 1 - \textstyle\sum_{i=1}^k \lambda_i \int_0^t w_{t-r}(a_i) p(r; x, a_i)\,\mathrm{d}r \\ &\quad + E_x^B \int_0^t \kappa\big(F(w_{t-r}(B_r)) - w_{t-r}(B_r)\big)\,\mathrm{d}r\,. \end{aligned}$$

Proof. Either the initial particle reaches age t before the time of the first branching, $\zeta^\emptyset > t$, or, vice versa, $\zeta^\emptyset$ attains some value s, $0 < s \leqslant t$. Split the expectation into two parts accordingly. It follows that

$$w_t(x) = E_x^B \left[\mathrm{e}^{-\sum_{i=1}^k \lambda_i \ell_t(a_i)} e^{-\kappa t} + \textstyle\int_0^t \kappa e^{-\kappa s} \mathrm{e}^{-\sum_{i=1}^k \lambda_i \ell_s(a_i)} F(w_{t-s}(B_s))\,\mathrm{d}s \right],$$

where ℓ_r is the local time up to time r of a standard Brownian motion. Write

$$H_r = \kappa r + \sum_{i=1}^k \lambda_i \ell_r(a_i)\,.$$

Then

$$w_t(x) = E_x^B \left[\mathrm{e}^{-H_t} + \textstyle\int_0^t \kappa\, \mathrm{e}^{-H_s} F(w_{t-s}(B_s))\,\mathrm{d}s \right].$$

Now use

$$\mathrm{e}^{-H_t} = 1 - \textstyle\int_0^t \mathrm{e}^{-(H_t - H_r)}\,\mathrm{d}H_r$$

to obtain

$$w_t(x) = 1 + E_x^B \textstyle\int_0^t \kappa\, F(w_{t-s}(B_s))\,\mathrm{d}s - R_x\,,$$

where

$$R_x = E_x^B \left[\textstyle\int_0^t \mathrm{e}^{-(H_t - H_r)}\,\mathrm{d}H_r + \int_0^t \kappa\, F(w_{t-s}(B_s)) \int_0^s \mathrm{e}^{-(H_s - H_r)}\,\mathrm{d}H_r\,\mathrm{d}s \right].$$

Hence,

$$\begin{aligned} R_x &= E_x^B \textstyle\int_0^t \left(\mathrm{e}^{-(H_t - H_r)} + \int_r^t \kappa\, \mathrm{e}^{-(H_s - H_r)} F(w_{t-s}(B_s))\,\mathrm{d}s \right) \mathrm{d}H_r \\ &= E_x^B \textstyle\int_0^t w_{t-r}(B_r)\,\mathrm{d}H_r \\ &= E_x^B \left[\textstyle\int_0^t \kappa\, w_{t-r}(B_r)\,\mathrm{d}r + \sum_{i=1}^k \lambda_i \int_0^t w_{t-r}(B_r)\,\mathrm{d}\ell_r(a_i) \right]. \end{aligned}$$

To complete the proof it remains to note

$$E_x^B \textstyle\int_0^t w_{t-r}(B_r)\,\mathrm{d}\ell_r(a_i) = \int_0^t w_{t-r}(a_i) p(r; x, a_i)\,\mathrm{d}r\,,$$

which holds since $\ell_\cdot(a_i)$ only increases on $\{t\,;\, B_t = a_i\}$, for $i = 1, \ldots, k$. $\square$

It follows from the preceding lemma and (1.3) that $u_t^{(n)}$ satisfies the equation

$$u_t^{(n)}(x) = \sum_{i=1}^k \lambda_i \textstyle\int_0^t (1 - u_{t-r}^{(n)}(a_i)/n)\, p(r; x, a_i)\,\mathrm{d}r - E_x^B \int_0^t \mathscr{A}\big(u_{t-r}^{(n)}(B_r)\big)\,\mathrm{d}r\,. \tag{4.7}$$

Next we remind that the branching exponent $\mathscr{A}$ is Lipschitz continuous in the sense that if $u_1, u_2 \leqslant C$ then there exists a constant $K(C)$ for which

$$\|\mathscr{A}(u_1) - \mathscr{A}(u_2)\| \leqslant K(C)\, \|u_1 - u_2\| . \tag{4.8}$$

For a proof, see Dawson (1993, Lemma 4.3.2).

Lemma 4.4. *For any $t > 0$ the limit function $\lim_{n\to\infty} u_t^{(n)}$ exists in supremum norm and is uniformly bounded. The limit is the unique solution v_t of the integral equation* (1.2).

Proof. For arbitrary $n_1, n_2 \geqslant 1$ we have by (4.7),

$$\begin{aligned}|u_t^{(n_1)}(x) - u_t^{(n_2)}(x)| &\leqslant \sum_{i=1}^{k} \lambda_i \int_0^t p(t-r;x,a_i) \left| \frac{u_r^{(n_1)}(a_i)}{n_1} - \frac{u_r^{(n_2)}(a_i)}{n_2} \right| \mathrm{d}r \\ &\quad + E_x^B \int_0^t \left| \mathscr{A}\left(u_{t-r}^{(n_1)}(B_r)\right) - \mathscr{A}\left(u_{t-r}^{(n_2)}(B_r)\right) \right| \mathrm{d}r .\end{aligned}$$

Due to (4.6) we can choose n_1, n_2 so large that $\sup_{r\geqslant 0} |u_r^{(n_1)}(a_i)/n_1 - u_r^{(n_2)}(a_i)/n_2|$ is less than some $\varepsilon > 0$. We can then use (4.6) again and combine with (4.8) to see that there is a constant K such that

$$|u_t^{(n_1)}(x) - u_t^{(n_2)}(x)| \leqslant \varepsilon \sum_{i=1}^{k} \lambda_i \sqrt{2/\pi}\, \sqrt{t} + K\, E_x^B \int_0^t \left| u_{t-r}^{(n_1)}(B_r) - u_{t-r}^{(n_2)}(B_r) \right| \mathrm{d}r .$$

The generalized Gronwall lemma by Dynkin (see e.g. Dawson (1993, Lemma 4.3.1), where the equivalent backward time version is given) now shows that

$$|u_t^{(n_1)}(x) - u_t^{(n_2)}(x)| \leqslant \varepsilon \sum_{i=1}^{k} \lambda_i \sqrt{2/\pi}\, \sqrt{t}\, \mathrm{e}^{Kt}, \quad t > 0,$$

and thus

$$\|u_t^{(n_1)} - u_t^{(n_2)}\| \to 0, \quad n_1, n_2 \to \infty .$$

Hence $u_t^{(n)}$, $n \geqslant 1$, is a uniform Cauchy sequence and we can let $\widehat{v}_t$ denote its limit. Clearly, for all t, $\|\widehat{v}_t\| < C$ in view of (4.6). Moreover, by (4.7) and Lebesgue's dominated convergence theorem, $\widehat{v}_t$ satisfies (1.2).

It was claimed in the introduction that a nonnegative solution of (1.2) is unique. We can now verify this in a simple way. Let $v_t^{(1)}$ and $v_t^{(2)}$ be two solutions of (1.2). Since

$$|v_t^{(1)}(x) - v_t^{(2)}(x)| \leqslant K\, E_x^B \int_0^t |v_{t-r}^{(1)}(B_r) - v_{t-r}^{(2)}(B_r)|\, \mathrm{d}r ,$$

another application of the generalized Gronwall lemma shows that $v^{(1)} \equiv v^{(2)}$. In the following we denote by v_t the uniformly bounded unique solution of (1.2). □

We remark that the previous lemma shows that for each fixed t the BBM local time $\ell_t^{(n)}$ converges in finite-dimensional distributions to the SBM local time L_t. For our further developments however, the more important conclusion is the uniform boundedness of v_t in t.

The next lemma is concerned with the rate of convergence of the log-Laplace function $u_t^{(n)}$ as both n and t tend to infinity.

Lemma 4.5. *Consider the constant C from* (4.6) *and let $K = K(C)$ be the corresponding Lipschitz constant such that* (4.8) *holds. Let t_n denote an increasing sequence of real numbers such that*

$$\frac{\sqrt{t_n}}{n}\, e^{K\,t_n} \to 0\,.$$

Then

$$\lim_{n\to\infty} \|u_{t_n}^{(n)} - v_{t_n}\| = 0\,.$$

Proof. By (1.2), (4.7) and Lemma 4.4,

$$\begin{aligned}
&|u_t^{(n)}(x) - v_t(x)| \\
&\quad\leqslant \frac{1}{n}\sum_{i=1}^{k}\lambda_i \int_0^t u_r^{(n)}(a_i)p(t-r;x,a_i)\,\mathrm{d}r + E_x^B\int_0^t \left|\mathscr{A}\left(u_{t-r}^{(n)}(B_r)\right) - \mathscr{A}\left(v_{t-r}(B_r)\right)\right| \mathrm{d}r \\
&\quad\leqslant \frac{C}{n}\sum_{i=1}^{k}\lambda_i \int_0^t p(r;0,0)\,\mathrm{d}r + K\,E_x^B\int_0^t \left|u_{t-r}^{(n)}(B_r) - v_{t-r}(B_r)\right| \mathrm{d}r\,.
\end{aligned}$$

As in the proof of Lemma 4.4 we obtain

$$\|u_t^{(n)} - v_t\| \leqslant C\,\sqrt{2/\pi}\,\frac{\sqrt{t}}{n}\,\mathrm{e}^{K\,t}\,,$$

and therefore

$$\|u_{t_n}^{(n)} - v_{t_n}\| \to 0\,,\quad n\to\infty\,,$$

by the assumption on t_n. □

4.3. The differential equation

In this subsection we show that the function v_t also solves a differential problem. This is then later used to show that its limit function $v = \lim_{t\to\infty} v_t$ is the unique solution of the problem (2.3).

Lemma 4.6. *For all $t > 0$ and $x \neq a_i$, $i = 1,\ldots,k$, the partial derivatives*

$$\frac{\partial}{\partial t}v_t(x)\,,\qquad \frac{\partial}{\partial x}v_t(x)\,,\qquad \frac{\partial^2}{\partial x^2}v_t(x)$$

exist and are continuous. The function $(t,x)\mapsto v_t(x)$ is a solution of the problem

$$\begin{aligned}
&\frac{\partial}{\partial t}v_t(x) = \frac{1}{2}\frac{\partial^2}{\partial x^2}v_t(x) - \mathscr{A}(v_t(x)) \qquad \text{on } \mathbb{R}\setminus\{a_1,\ldots,a_k\}\,,\\
&\frac{\partial}{\partial x}v_t(a_i-) - \frac{\partial}{\partial x}v_t(a_i+) = 2\lambda_i\,, \qquad \text{for all } t>0 \text{ and } i=1,\ldots,k\,,\\
&v_0(x) = 0\,.
\end{aligned} \tag{4.9}$$

Proof. Let

$$G(t,x) := E_x^B \int_0^t \mathscr{A}\big(v_{t-r}(B_r)\big)\,\mathrm{d}r\,.$$

Because $u \mapsto \mathscr{A}(u)$ is continuous and $(t,x) \mapsto v_t(x)$ is bounded and measurable it follows (see e.g. Durrett, 1984, p. 226) that the function $(t,x) \mapsto G(t,x)$ is continuous and $\partial G/\partial x$ exists and is jointly continuous. Consequently, from (1.2), $(t,x) \mapsto v_t(x)$ is continuous. Further, $(\partial/\partial x)v_t(x)$ exists and is jointly continuous for every $t > 0$ and $x \neq a_i$. The claimed condition for the first derivatives at $x = a_i$ follows from the elementary fact that

$$\frac{\partial}{\partial x}\int_0^t p(r;x,a_i)\,\mathrm{d}r = \begin{cases} -P_{|x-a_i|}(\tau_0 < t), & \text{for } x > a_i \\ P_{|x-a_i|}(\tau_0 < t), & \text{for } x < a_i\,, \end{cases}$$

where $\tau_0 := \inf\{t\,;\, B_t = 0\}$. Let $N > 0$ be such that $-N < a_1$. Then, by the continuity of $\partial G/\partial x$ it is easily seen that there exists a constant C_N such that

$$\left|\frac{\partial}{\partial x}v_t(x)\right| < C_N \tag{4.10}$$

for all $0 < t < N$ and $-N < x < a_1$. Clearly, also

$$\left|\frac{\partial}{\partial x}v_t(a_1-)\right| < C_N\,.$$

Proceeding recursively gives us a constant also denoted by C_N such that (4.10) holds for all $0 < t < N$ and $-N < x < N$. It follows that

$$|v_t(x) - v_t(y)| < C_N\,|x-y|$$

for all $0 < t < N$ and $-N < x < N$. Because $u \mapsto \mathscr{A}(u)$ is Lipschitz continuous we argue as in Durrett (1984, pp. 226–227), that $\partial^2 G/\partial x^2$ and $\partial G/\partial t$ exist and are jointly continuous. Moreover, G satisfies

$$\frac{\partial}{\partial t}G(t,x) = \frac{1}{2}\frac{\partial^2}{\partial x^2}G(t,x) + \mathscr{A}(v_t(x))\,.$$

Straightforward computations now show that $v_t(x)$ solves for $t > 0$ and $x \neq a_i$ the PDE in (4.9). The initial condition being obvious the proof of the lemma is complete. □

Let

$$v := \lim_{t\to\infty} v_t\,,$$

which is well-defined due to monotonicity.

Lemma 4.7. *The limit function v is continuous and convex in the intervals (a_i, a_{i+1}), $i = 0,\ldots,k$, and*

$$\|v_t - v\| \to 0\,, \quad t \to \infty\,.$$

Proof. We show first that there exists a constant C such that

$$\sup_{t\geqslant 0} \|\frac{\partial}{\partial x}v_t\| \leqslant C. \tag{4.11}$$

In fact, since $t \mapsto v_t(x)$ is nondecreasing it is obvious from the PDE in (4.9) that for every fixed $t > 0$, the function $x \to v_t(x)$ is convex in the intervals (a_i, a_{i+1}), $i = 0, \ldots, k$. From this and the fact that $v_t(x)$ is bounded it is seen that

$$\lim_{x\to\pm\infty} v_t(x) = \lim_{x\to\pm\infty} \frac{\partial}{\partial x}v_t(x) = 0.$$

Now we can argue that the solution v_t of (4.9) satisfies (4.11) exactly as in the proof of Lemma 4.1 where it was shown that the derivative of the corresponding solution g was uniformly bounded.

By (4.11) the family $\{v_t\,;\, t\geqslant 0\}$ is equicontinuous: there exists a constant C such that for all x, y and $t\geqslant 0$

$$|v_t(x) - v_t(y)| < C\,|x-y|.$$

Consequently, $\|v_t - v\| \to 0$ and the continuity and convexity of v_t carries over to v. □

Lemma 4.8. *The function v is a solution of the problem* (2.3).

Proof. We begin with

$$\lim_{t\to\infty} \|\frac{\partial}{\partial t}v_t\| = 0. \tag{4.12}$$

Note that $t \mapsto G(t,x)$ is nondecreasing. Indeed, letting $t_1 < t_2$ we have

$$E_x^B \int_0^{t_2} \mathscr{A}\big(v_{t_2-r}(B_r)\big)\,\mathrm{d}r \geqslant E_x^B \int_0^{t_1} \mathscr{A}\big(v_{t_2-r}(B_r)\big)\,\mathrm{d}r \geqslant E_x^B \int_0^{t_1} \mathscr{A}\big(v_{t_1-r}(B_r)\big)\,\mathrm{d}r,$$

since $u \mapsto \mathscr{A}(u)$ and $t \mapsto v_t(x)$ are positive and non-decreasing. Therefore, $\partial G/\partial t \geqslant 0$. Combined with (1.2) this gives

$$\frac{\partial}{\partial t}v_t(x) \leqslant \sum_{i=1}^{k} \lambda_i\, p(t;x,a_i) \leqslant c\,t^{-1/2},$$

which proves (4.12).

Now we can prove that $v \in C^2(-\infty, a_1)$ and solves the claimed ODE in $(-\infty, a_1)$. Multiplication of the PDE in (1.2) with $\partial v_t/\partial x$ and an integration from $-\infty$ to $x < a_1$ lead to

$$\int_{-\infty}^{x} \frac{\partial}{\partial x}v_t(y)\, \frac{\partial}{\partial t}v_t(y)\,\mathrm{d}y = \frac{1}{2}\Big(\frac{\partial}{\partial x}v_t(x)\Big)^2 - \int_0^{v_t(x)} \mathscr{A}(u)\,\mathrm{d}u.$$

By (4.11) and (4.12) the left-hand side tends to zero as $t \to \infty$. Hence for every $x < a_1$,

$$\lim_{t\to\infty} \frac{1}{2}\Big(\frac{\partial}{\partial x}v_t(x)\Big)^2 = \int_0^{v(x)} \mathscr{A}(u)\,\mathrm{d}u.$$

Because the right-hand side is continuous,

$$v'(x) = \lim_{t\to\infty} \frac{\partial}{\partial x} v_t(x)$$

exists and satisfies

$$\tfrac{1}{2}\left(v'(x)\right)^2 = \int_0^{v(x)} \mathscr{A}(u)\,\mathrm{d}u .$$

Letting $t \to \infty$ in the PDE of (4.9) and using (4.12) it follows that for every $x < a_1$,

$$\lim_{t\to\infty} \frac{1}{2}\frac{\partial^2}{\partial x^2} v_t(x) = \mathscr{A}(v(x)) .$$

Consequently, v'' exists and satisfies

$$\frac{1}{2} v''(x) = \mathscr{A}(v(x)) . \tag{4.13}$$

Assume next that $a_1 < x < a_2$. Because v is convex there exists a point b_1 such that $a_1 \leqslant b_1 \leqslant a_2$ and v is monotone in (a_1, b_1) and (b_1, a_2). Intuitively one expects that, in fact, $a_1 < b_1 < a_2$ in the case $a_2 < \infty$. However, this property is not needed for our argument. Without loss of generality, suppose that v is (strictly) decreasing in (a_1, x) and, hence, every v_t has the same property. Proceeding as above,

$$\lim_{t\to\infty} \left(\frac{1}{4}\left(\frac{\partial}{\partial x} v_t(x)\right)^2 - \frac{1}{4}\left(\frac{\partial}{\partial x} v_t(a_1+)\right)^2 \right) = \int_{v(a_1)}^{v(x)} \mathscr{A}(u)\,\mathrm{d}u .$$

Since

$$\lim_{t\to\infty} \frac{\partial}{\partial x} v_t(a_1+) = v'(a_1-) - \lambda_1 ,$$

it follows that $v'(x)$ exists and, further, by continuity

$$v'(a_1-) - v'(a_1+) = \lambda_1 .$$

The ODE (4.13) is obtained similarly as above. The proof is now easily completed by proceeding recursively. □

Having verified that the problem (2.3) has a solution it remains to prove uniqueness.

Lemma 4.9. *The differential problem* (2.3) *has at most one solution.*

Proof. Suppose g is a solution. For the almost identical system in Lemma 4.1 it was already shown that the solution and its first derivative are bounded. The proof is even simpler in this case. Hence $\|g'\|$ and $\|g\|$ are finite.

Let g_1 and g_2 denote two solutions of (2.3). By the previous argument we can assume that they are bounded. We split the proof into two parts. Recall that we have put $a_0 = -\infty$ and $a_{k+1} = \infty$.

(1) Take x_1 and x_2 such that

$$a_i \leqslant x_1 \leqslant x_2 \leqslant a_{i+1} \quad \text{for some } i,\ 0 \leqslant i \leqslant k .$$

Suppose

$$g_1(x_1) = g_2(x_1), \quad g_1(x_2) = g_2(x_2), \quad g_1(x) \geqslant g_2(x) \text{ for all } x \in (x_1, x_2).$$

Then

$$\begin{aligned} 0 &= \int_{x_1}^{x_2} g_1(\tfrac{1}{2}g_2'' - \mathscr{A}(g_2))\,\mathrm{d}x - \int_{x_1}^{x_2} g_2(\tfrac{1}{2}g_1'' - \mathscr{A}(g_1))\,\mathrm{d}x \\ &= \tfrac{1}{2}\int_{x_1}^{x_2}(g_1 g_2'' - g_2 g_1'')\,\mathrm{d}x + \int_{x_1}^{x_2}\Big(g_2\mathscr{A}(g_1) - g_1\mathscr{A}(g_2)\Big)\,\mathrm{d}x. \end{aligned}$$

This is justified since e.g.

$$\int_a^\infty g_2\mathscr{A}(g_1)\,\mathrm{d}x \leqslant C_1 \int_a^\infty g_1''\,\mathrm{d}x = -C_1 g'(a) \leqslant C_2, \quad a \geqslant a_k,$$

for some constants C_1, C_2. Using integration by parts,

$$\int_{x_1}^{x_2}(g_1 g_2'' - g_2 g_1'')\,\mathrm{d}x = g_1(x_2)(g_2'(x_2-) - g_1'(x_2-)) + g_1(x_1)(g_1'(x_1+) - g_2'(x_1+)) \geqslant 0$$

by assumption. We also have

$$\int_{x_1}^{x_2}\big(g_2\mathscr{A}(g_1) - g_1\mathscr{A}(g_2)\big)\,\mathrm{d}x = \int_{x_1}^{x_2} g_1 g_2\Big(\frac{\mathscr{A}(g_1)}{g_1} - \frac{\mathscr{A}(g_2)}{g_2}\Big)\,\mathrm{d}x \geqslant 0,$$

since the assumption on the Lévy measure ν for $\mathscr{A}$ implies that $\mathscr{A}(u)/u \leqslant \mathscr{A}'(u) < \infty$ from which it follows that $\mathscr{A}(u)/u$ is monotone. Thus $g_1 = g_2$ in $[x_1, x_2]$ and hence, using the conditions at the corner points $a_1, \ldots, a_k$, we have $g_1 = g_2$ in $\mathbb{R}$.

(2) Now assume there is a number x_1, possibly $x_1 = \infty$, such that

$$g_1(x_1) = g_2(x_1), \qquad g_1(x) \geqslant g_2(x) \quad \text{in} \quad (-\infty, x_1).$$

For some $0 \leqslant i \leqslant k$ we have $x_1 \in [a_i, a_{i+1}]$. Similarly as in case (1), perform integration by parts over $(-\infty, x_1]$. Then use the boundary conditions to get

$$0 = \sum_{j=1}^{i} \lambda_j(g_1(a_j) - g_2(a_j)) + \int_{-\infty}^{x_1} g_1 g_2\Big(\frac{\mathscr{A}(g_1)}{g_1} - \frac{\mathscr{A}(g_2)}{g_2}\Big)\,\mathrm{d}x \geqslant 0.$$

Hence g_1 and g_2 must coincide in $(-\infty, x_1]$. □

4.4. *Proof of the diffusion approximation*

In the preceding sections we have obtained existence and uniqueness of a solution to (2.3). Choose a sequence t_n which satisfies the assumption in Lemma 4.5. By Lemmas 4.2, 4.5 and 4.7 the three terms on the right side of (1.6) converge to zero as $n \to \infty$. Hence $u^{(n)}$ converges uniformly to v. This completes the proof of the theorem.

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